Exchange economy with indirect utility functions

Consider an exchange economy with two consumers A and B, and two goods 1 and 2. Each consumer has an income m^j, $j = A, B$. Suppose that the indirect utility functions of both consumers are:

$$v^A(p_1, p_2, m^A) = \ln m^A - \frac{1}{2}\ln p_1 - \ln p_2$$
$$v^B(p_1, p_2, m^B) = (p_1^{-1} + p_2^{-1})m^B.$$

Each consumer has an initial endowment of 6 units of good 1 and 2 units of good 2. Calculate the relative competitive equilibrium price in this economy, assuming that good 1 is the numéraire ($p_1 = 1$).

Solution

We use Roy's identity to obtain the Marshallian demands:

$$x_1^A(p, m^A) = -\frac{\partial v^A/\partial p_1}{\partial v^A/\partial m^A} = -\frac{(-1/2)/p_1}{1/m^A} = \frac{m^A}{2p_1}$$

$$x_1^B(p, m^B) = -\frac{\partial v^B/\partial p_1}{\partial v^B/\partial m^B} = -\frac{-m^B/p_1^2}{p_1^{-1} + p_2^{-1}} = \frac{m^B p_1^{-2}}{p_1^{-1} + p_2^{-1}}$$

The income of the consumers is defined as the value of the initial endowments. That is,

$$m^A = m^B = m = 6p_1 + 2p_2.$$

The Walras law tells us that if one of the markets is in equilibrium, the other will also be in equilibrium. Let us consider the market for good 1. The equilibrium in the market for good 1 can be characterized by the aggregate excess demand function:

$$Z_1(p) = e_1^A(p) + e_1^B(p) = 0$$

where p represents the price system, and e_1^j represents the net demand for good 1 by consumer j, i.e., the difference between the demand for good 1, $x_1^j(p, m)$, by consumer j and their initial endowment of good 1.

Substituting the values of the Marshallian demands, the initial endowments, and the income into the aggregate excess demand function for good 1, we obtain (recalling that $p_1 = 1$):

$$Z_1(p) = x_1^A - 6 + x_1^B - 6$$

$$Z_1(p) = \left(\frac{m^A}{2p_1} - 6\right) + \left(\frac{m^B p_1^{-2}}{p_1^{-1} + p_2^{-1}} - 6\right) = \left(\frac{m}{2 \cdot 1} - 6\right) + \left(\frac{m \cdot 1^{-2}}{1^{-1} + p_2^{-1}} - 6\right)$$
$$= \frac{6 + 2p_2}{2} + \frac{6 + 2p_2}{1 + 1/p_2} - 12 = 3 + p_2 + \frac{6 + 2p_2}{1 + 1/p_2} - 12 = p_2 - 9 + \frac{6 + 2p_2}{1 + 1/p_2} =$$
$$= \frac{p_2 + 1 - 9 - 9/p_2 + 6 + 2p_2}{1 + 1/p_2} = \frac{-2 - 9/p_2 + 3p_2}{1 + 1/p_2} = \frac{3p_2^2 - 2p_2 - 9}{p_2 + 1}$$

Then, as we know that $Z_1(p) = 0$

$$\frac{3p_2^2 - 2p_2 - 9}{1 + p_2} = 0.$$

Solving this equation for p_2, we obtain

$$p_2^* = \frac{1 + 2\sqrt{7}}{3} \approx 2.0972.$$

Therefore, the relative equilibrium price is

$$\frac{p_2^*}{p_1^*} = \frac{1 + 2\sqrt{7}}{3}.$$